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Attractors in fully asymmetric neural networks

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Abstract. The statistical properties of the length of the cycles and of the weights of the attraction basins in fully asymmetric neural networks (i.e. with completely uncorrelated synapses) are computed in the framework of the annealed approximation which we previously introduced for the study of Kauffman networks. Our results show that this model behaves essentially as a random map possessing a reversal symmetry. Comparison with numerical results suggests that the approximation could become exact in the large-size limit.

1. Introduction

In the past decade attractor neural networks have been the subject of an intense study as a model of associative memory. The 'ancestor' of these models, the Hopfield model [1], was defined as follows. There is a set of N neurons, each one associated with a binary variable $\sigma_i \in \{0, 1\}, i \in \Omega = \{1, \ldots, N\}$, representing its activity. The synaptic couplings between these model neurons, J_{ij}, are chosen at random at the beginning and kept fixed, and then the system evolves deterministically according to the equation

$$\sigma_i(t+1) = \mathrm{sign}\left(\sum_j J_{ij}\sigma_j(t) \right) \tag{1}$$

(parallel updating; alternatively, one can consider sequential updating when $\sigma_i(t+1)$ is determined by the state of $\sigma_j(t+1)$ for $j < i$ and by $\sigma_j(t)$ for $j > i$).

This procedure defines a disordered dynamical system: the evolution is deterministic, but its rules are chosen at random at the beginning and kept fixed. In other words, we can rewrite the dynamic law in the form

$$\mathcal{C}(t+1) = f_J(\mathcal{C}(t)) \tag{2}$$

where $\mathcal{C}$ represents a configuration of the system, i.e. a set of values of the N variables σ_i, f_J is a random realization of a deterministic map and the set of indices J labels the realization of the dynamic rules.

The most natural distance in configuration space is the normalized Hamming distance, defined as

$$d(C, C') = \frac{1}{N} \sum_i |\sigma_i - \sigma_i'|. \tag{3}$$

We are interested in the statistical properties of the motion asymptotically in time and system size. As the motion is deterministic and configuration space is finite, asymptotically in time the dynamics takes place on periodic orbits, and the quantities of interest are the

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lengths and number of such orbits as well as the size of their attraction basins. Such quantities are random variables, depending on the realization of the dynamical rules, and we will study their probability distribution.

When the couplings are symmetric ($J_{ij} = J_{ji}$) it is possible to define a Hamiltonian so that equation (1) represents the zero-temperature dynamics of a thermodynamic system. In particular, if the J_{ij} are chosen from a distribution with zero mean and variance $1/N$ (for instance a Gaussian distribution) we are dealing with the zero-temperature dynamics of the SK model (in the case of sequential updating: the parallel updating does not imply a relaxational dynamics). In the Hopfield model the couplings are also symmetric, but they are chosen according to the Hebbian rule:

$$J_{ij} = \sum_{\mu=1}^{P} \xi_i^\mu \xi_j^\mu \tag{4}$$

where the P vectors of N binary variables, ξ^μ, represent the memorized patterns. The system is able to memorize, in the sense that the patterns are fixed points of the dynamics and they are stable against random perturbations if their number does not exceed the capacity of the network, i.e. if P is not larger than $\alpha_c N$, with $\alpha_c \approx 0.14$. So, given a number of microscopic states growing as 2^N, the Hopfield model is able to memorize a number of patterns growing linearly with N.

Asymmetric neural networks received much attention in the literature in the late 1980s [3–8]. In 1986 a generalization of the Hopfield model was proposed by taking into account also asymmetric couplings [2]. This generalization appears more realistic, since synapses in nature are in general not symmetric, and it suggests a possible way to distinguish between a network that has remembered a learnt pattern and a network which is in a confused state (such a distinction is not possible in the Hopfield model). In fact, in asymmetric neural networks, two kind of attractors are present: 'ordered' attractors, that are either short cycles or fixed points, and 'chaotic' attractors, whose length grows exponentially with system size. The first numerical observations of this twofold nature of the attractors are due to Gutfreund *et al* [5] and Nützel [9].

In this paper we are mainly interested in the study of the properties of the attractors, such as the probability distributions of their lengths, number and size of their attraction basins. We will only consider the case of fully asymmetric couplings, i.e. J_{ij} and J_{ji} are independent random variables. In this case analytical results have already been obtained about the correlation functions [6, 7] and about the number of attractors [11], but more about the attractors can be said by using a simple stochastic scheme based on the annealed approximation. This approximation was introduced in the study of disordered dynamical systems by Derrida and Pomeau [12] to study damage spreading in Kauffman networks (a disordered dynamical system proposed as a model of the genetic regulation in cells [13]). In [14] we showed that it can also be used to obtain information about the attractors of that model.

A reason of interest of this study is that asymmetric neural networks are the limit case of a one-parameter family of models, the parameter η representing the symmetry of the synaptic couplings:

$$\eta = \frac{\langle J_{ij} J_{ji} \rangle}{\langle J_{ij} \rangle^2}. \tag{5}$$

The case $\eta = 0$ represents the present model (fully asymmetric couplings), while for $\eta = 1$ the couplings are fully symmetric and we obtain the mean-field model of spin glasses.

Thus, the parameter η connects with continuity asymmetric neural networks to a disordered system of statistical mechanics.

It was suggested through numerical simulations that the model with generic correlation undergoes a dynamical transition when η is changed [9, 10, 15]. The transition seems to take place when the absolute value of η crosses the value $\frac{1}{2}$. For $|\eta| < \frac{1}{2}$ the dynamics is chaotic and the typical length of the cycles increases exponentially with the number of neurons N, while for $|\eta| > \frac{1}{2}$ the dynamics is frozen and the typical length of the cycles does not increase with system size (most of the cycles have a length of 2, for positive η, and 4 for negative η).

This transition is reminiscent of the dynamical transition taking place in Kauffman networks. Also in that case the typical length of the cycles grows exponentially with N in the so-called chaotic phase, remains finite in the frozen phase and grows less than exponentially with N on the critical line [13, 14]. It was claimed by Kauffman that the critical line of his model can be a good model of the genetic regulatory systems acting in cell differentiation, thus showing that such systems do not need to be tuned in the very details by natural selection but behave similarly to typical realizations of an ensemble of random regulatory networks [13]. It is possible that, analogously, also the supposed critical point in attractor neural networks, where chaotic and ordered cycles coexist, can suggest something interesting from a biological point of view. We think that our method can be modified to give information about systems with generic asymmetry and about the supposed phase transition that they undergo, though this probably requires going beyond the annealed approximation.

2. Closing probabilities

2.1. General framework

Our strategy for the study of attractors in disordered dynamical systems has as its starting point the probability distribution of the distance at different time steps. Actually, the information contained in the distribution of the distance is much more than what we need and this distribution is in principle a very complicated object, so that our approach may seem to complicate the problem. But, in some cases, the distance can be well approximated by a suitably defined stochastic process and the computation becomes much easier. The simplest possibility is to approximate the distance with a Markovian stochastic process. This is what we call here the *annealed approximation*.

An apparently paradoxical aspect of this approach is that in disordered dynamical systems attractors exist due to the fact that the motion is deterministic. Stochastic processes, on the other hand, have nothing similar to a limit cycle. Nevertheless, all the properties of the attractors can be derived from the distribution of distances, which is a well defined object in both kinds of models. We cannot pursue this analogy up to times larger than the time of first recurrence of a configuration already visited, when the deterministic motion becomes periodic. But this is enough, since the first recurrence provides us with every information about the length of the cycles and the transient time.

The fundamental object of our study will then be the distribution of distances between configurations at time steps t and $t' > t$ on the same trajectory, restricted to trajectories that have not yet visited twice any configuration up to the larger time t' (thus the effects of periodicity do not yet appear). We will call this condition the opening condition, and denote it by the symbol $A_{t'}$. The closing probability $\pi_N(t, t')$ is the probability that configurations

at time steps t and t' are equal ($d(t,t')=0$), subject to the opening condition:

$$\pi_N(t,t') = \Pr\{d(t,t')=0|A_{t'}\} \tag{6}$$

(the subscript N is there to remind us of the dependence on system size).

After the closing time t' the trajectory enters a periodic orbit of length $l = t' - t$, where t is the transient time. In terms of the closing probabilities, the probability to find such a trajectory is easily computed. First we have to know the probability $F_N(t)$ that the trajectory was not closed before time $t' = t + l$. This obeys the equation $F_N(t+1) = F_N(t)(1 - \sum_{t'=0}^{t-1} \pi_N(t',t))$, whence, introducing a continuous time variable, we obtain

$$F_N(t) = \exp\left(- \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \, \pi_N(t',t'') \right) \tag{7}$$

(to have a slightly simpler formula we made the hypothesis that the typical closing times are long, which is normally the case in the chaotic phase, where they grow exponentially with system size N, and we transformed the sum into an integral).

The probability of finding a trajectory that, after a transient time t, enters a cycle of length l is then obtained by multiplying $F_N(t+l)$ by the closing probability $\pi_N(t,t+l)$.

2.2. The annealed approximation

Regarding the distance as a Markovian stochastic process is a very drastic approximation. In our case it sounds reasonable when the temporal distance l is large, since the model that we study is known to have a behaviour very reminiscent of chaos [15]. However, in this way we neglect some memory effects, which can play a fundamental role in systems with non-zero symmetry.

This approximation was first used in this context by Derrida and Pomeau [12], who studied the damage spreading in Kauffman networks. They showed that the average value of the Hamming distance between two different trajectories is equivalent, in the infinite-system limit, to the average value of the Markovian stochastic process obtained by extracting new dynamical rules at every time step and remembering only the value of the distance at time step $t-1$. Thus, the disorder is treated as annealed rather than as quenched. In other words, instead of considering an ensemble of trajectories, each one taking place on a fixed realization of the dynamical rules, they consider an ensemble of trajectories moving from one realization of the dynamical rules to another one, in the same spirit in which the annealed average is used for disordered thermodynamical systems.

The above procedure can be shown to exactly describe the evolution of the average distance up to a time of order $\log N$ in disordered systems with finite connectivity [16, 17], but we think that its validity is more general. In systems with infinite connectivity such as the one that we are studying here, or when one is interested in the whole distribution of the distance, the equivalence between the two dynamics has not been proved, and we have to assume that a typical trajectory of the quenched system loses memory of the details of the realization of dynamical rules under which it evolves. In the random map model [18] this is trivially true. In other cases this can be thought of as a maximal ignorance hypothesis, whose consequences must then be compared with numerical simulations.

Let us state some of these consequences. A Markovian stochastic process, if its transition probability is ergodic†, converges to a stationary stochastic variable independent of the initial

† In the present case, in order to have an ergodic transition probability, we must first exclude the distance $d=0$ which is an absorbing point (if $d(t,t+l)=0$, we must have with a probability of 1 $d(t',t'+l)=0$ for every $t' \geqslant t$), that is we have to impose the condition that the trajectory is not yet closed, as we did.

distribution. This means that the closing probability $\pi_N(t, t+l)$ converges to a stationary value π_N^*. This is also independent of l if the transition probability does not depend on this quantity. We will show that this happens in the present case, at least for l large enough.

It is then easy to compute the probability of a trajectory which, after a transient time t, enters a cycle of length l (with l and t large enough, so that the closing probability has reached its asymptotic value): using the results of last section, we obtain

$$\Pr\{T = t, L = l\} = \frac{1}{\tau_N^2} \exp\left(-\frac{1}{2}\left(\frac{t+l}{\tau_N}\right)^2\right) \tag{8}$$

where $\tau_N = {\pi_N^*}^{-1/2}$ is the typical timescale of the problem, in the sense that the random variable t/τ_N has a well defined density of probability even in the limit where τ goes to infinity. All the dependence on system size is contained in the factor π_N^*, which is expected to decrease exponentially with N in the chaotic phase. For instance, for a uniform random map [18], which is the most chaotic disordered dynamical system, it holds that $\pi_N^* = 1/2^N$, and consequently the typical timescale of the attractors grows as $2^{N/2}$.

The properties of random maps can be easily generalized starting from equation (8). An interesting quantity is the distribution of the attraction basin weights. This was analytically computed by Derrida and Flyvbjerg [19] for the case of the uniform random map. The weight of the attraction basin of cycle α, W_α, is defined as the probability to extract at random an initial configuration which will asymptotically reach the attractor α. The statistical information about the distribution of the weights can be expressed through the 'moments' $\langle Y_n \rangle$, defined as

$$\langle Y_n \rangle = \sum_\alpha \langle W_\alpha^n \rangle. \tag{9}$$

$Y_1 = 1$ due to the normalization of the weights and Y_2 represents the average weight in a given dynamical system. Its extreme values, 1 and 0, correspond respectively to the 'ergodic' case where there is only one relevant attractor and to the case where there is an infinite number of relevant attractors, while a finite value of Y_2 means that there is a finite number of attractors with non-vanishing weight. This quantity fluctuates from sample to sample, so it is necessary to consider an average over the realizations of the dynamical rules, that is represented by the angular brackets.

The method used in [19] to compute the distribution of the weight can be applied without modifications to all disordered dynamical systems where the closing probability reaches an asymptotic value, π_N^*, and the results do not depend on this value in the large-size limit. Thus the distribution of the attraction basin weights is *universal* for all the disordered dynamical systems where the closing probability reaches a stationary value [14, 21], apart for systems which possess some symmetry. The result for the average value of the Y_n is [19]

$$\langle Y_n \rangle = \frac{4^{n-1}[(n-1)!]^2}{(2n-1)!}. \tag{10}$$

The fluctuations from sample to sample can also be computed. For example the fluctuations of Y_2 are measured by $\langle Y_2^2 \rangle - \langle Y_2 \rangle^2$, and do not cancel even in the infinite-size limit $N \to \infty$.

The average number of attractors of length l can be computed starting with the relation

$$\langle n_a(l) \rangle = \frac{2^N}{l} \Pr\{T = 0, L = l\}. \tag{11}$$

In this formula, the probability $\Pr\{L = l, T = 0\}$ should be computed by multiplying $F_N(l)$, given in equation (7), by the closing probability $\pi_N(0, l)$. This is different from the asymptotic value (for large t) of $\pi_N(t, t + l)$. According to the hypothesis, on which the annealed approximation relies, that the system is going to forget the details of the evolution, we expect no correlations between the initial configuration and a configuration at a large time l or, in other words, we expect the closing probability to be, asymptotically in l, $\pi_N(0, l) = 1/2^N$. For chaotic Kauffman networks this can be explicitly computed in the framework of the annealed approximation, which is then consistent under this point of view. Moreover, in this case we find $\pi_N(0, l) = c_l 1/2^N$, where c_l does not depend on N. Thus it holds

$$\langle n_a(l) \rangle \approx \frac{c_l}{l} \exp(-l^2/2\tau^2). \tag{12}$$

Summing over l we obtain the average value of the total number of cycles, whose leading term in N is equal to $\log \tau_N$. Since in the chaotic phase the timescale grows exponentially with system size, the number of attractors is proportional to N in this case. This computation holds for chaotic Kauffman networks in the framework of the annealed approximation, but we expect it to hold more generally under the hypothesis discussed above.

2.3. Master equation

The general scheme described above must be modified in the case studied in this paper, to take into account the symmetry of the problem. Let us define the reversal operator, R, which reverses all the spins. This operator commutes with the dynamics of the system. Using the notation defined in equation (2), we can write:

$$f_J(R\mathcal{C}) = Rf_J(\mathcal{C}). \tag{13}$$

This implies that we can define two different closing times.

(1) The first time t when $\mathcal{C}(t)$ is equal to $\mathcal{C}(t + l)$.

(2) The first time when $\mathcal{C}(t + l)$ is equal to $R\mathcal{C}(t)$: then equation (13) implies $\mathcal{C}(t + 2l) = \mathcal{C}(t)$. In other words, the trajectory has reached, after a transient time t, a cycle of length $2l$.

These closing events can be described in terms of the Hamming distance between configurations: the first one corresponds to $d(t, t+l) = 0$, while the second one corresponds to $d(t, t + l) = 1$. Thus two closing probabilities must be defined:

$$\pi_N^{(0)}(t, t') = \Pr\{d(t, t') = 0 | A_{t'}\} \tag{14}$$

$$\pi_N^{(1)}(t, t') = \Pr\{d(t, t') = 1 | A_{t'}\} \tag{15}$$

and the opening condition, A_t, has the meaning that up to time t it never occurred for either $d(t_1, t_2) = 0$ or $d(t_1, t_2) = 1$. Our task is now to compute the master equation for the distribution of the distance under the opening condition (this means that we consider only trajectories not yet closed) and under the hypothesis that the distribution of $d(t + 1, t' + 1)$ depends only on the distribution of $d(t, t')$. This is not a difficult task. To simplify the formulae slightly we will consider, instead of the distance, the overlap $q = 1 - d$, which is measured by the number of elements whose state is the same in the two configurations, divided by N.

An element σ_i is in the same state at time $t+1$ and $t'+1$ if its local field has the same sign at time steps t and t'. Thus it holds

$$\sigma_i(t+1)\sigma_i(t'+1) = \mathrm{sign}\left(\sum_{jk} J_{ij}J_{ik}\sigma_k(t)\sigma_j(t')\right). \tag{16}$$

Let us consider separately the contribution to this sum coming from the spins whose state is the same at time steps t and t', whose number is $Nq(t,t')$, and that belong to a set that we indicate with the name $I(t,t')$. We can then write

$$\sigma_i(t+1)\sigma_i(t'+1) = \mathrm{sign}((h_i^+(t))^2 - (h_i^-(t))^2) \tag{17}$$

where

$$\begin{aligned} h_i^+(t) &= \sum_{j\in I(t,t')} J_{ij}\sigma_j(t) \\ h_i^-(t) &= \sum_{j\in \Omega/I(t,t'))} J_{ij}\sigma_j(t). \end{aligned} \tag{18}$$

The annealed approximation consists of considering the local fields as random variables, correlated to the previous story of the system only through the value of $q(t,t')$. In this spirit, we consider a dynamics in which the local fields are extracted at random at every time step, under the following assumptions.

(1) The local fields at different points are independent random variables.

(2) The value of $\sigma_j(t)$ is independent on the synaptic coupling J_{ij}.

Both of these assumptions encounter troubles when the synaptic couplings are correlated with each other, but they are quite reasonable for $\eta = 0$, which is the case that we are studying now. Assumption (1) implies that the transition probability is a binomial one:

$$\Pr\{q(t+1,t'+1) = q_n | q(t,t') = q_m\} = \binom{N}{n}(\gamma(q_m))^n(1-\gamma(q_m))^{N-n} \tag{19}$$

where $q_n = n/N$, and $\gamma(q)$ is the probability that $|\tilde{h}^+(q)| > |\tilde{h}^-(1-q)|$, where, following assumption (2), $\tilde{h}^{\pm}(q)$ is a Gaussian variable with mean value zero and variance q (this result is independent on the details of the distribution of the couplings, provided that they are all independent variables with mean value zero and with the same variance). A straightforward computation shows that

$$\gamma(q) = \frac{2}{\pi}\arcsin\sqrt{q}. \tag{20}$$

The Markov process associated to this transition probability is ergodic if we exclude the values $q = 0$ and $q = 1$ as starting points, as we do imposing the opening condition, and the distribution of the distance evolves towards a stationary distribution. Moreover, since the transition probability is independent on $l = t' - t$, also the stationary distribution is independent on l, which only appears in the initial distribution of the variable $q(0,l)$. It is also evident from the symmetry of the problem that it must hold $\gamma(q) = 1 - \gamma(1-q)$, so, if the initial distribution is also symmetric (e.g. a binomial distribution around $q = \frac{1}{2}$), the overlap distribution will be symmetric at every time step and it will be concentrated around the value $Q(t,t') = D(t,t') = \frac{1}{2}$ (the distributions of the overlap and of the distance are perfectly equivalent in this case). The stationary distribution is, independently on the initial one, concentrated around the value Q^* solution of the self-consistent equation:

$$Q^* = \gamma(Q^*) = \frac{2}{\pi}\arcsin\sqrt{Q^*}. \tag{21}$$

This equation has three solutions: $\frac{1}{2}$, 1 and 0, but only the first one can be accepted, according to the criterion $|\gamma'(Q^*)| < 1$, which can be obtained either as the stability condition of the fixed point Q^* of the map $Q(t+1,t'+1) = \gamma(Q(t,t'))$, or as the condition that the variance of the stationary distribution is positive (see equation (25) below).

Equation (21) is equivalent to the equation for the stationary value of the correlation function rigorously derived in [6] through a functional integral approach, so that one can see from this comparison that the annealed approximation gives an exact (though trivial) result concerning the average overlap. But our task here is to compute the whole stationary distribution of the overlap, and we cannot prove that the annealed approximation is correct to this extent, so we have to rely upon simulations to control its validity.

Although it is concentrated around $Q^* = \frac{1}{2}$, the stationary distribution is much broader than a binomial one and thus the closing probability is exponentially larger than $1/2^N$. In order to compute its value, we proceed in this way [14]. Since the transition probability is exponentially concentrated, we look for a solution of the form:

$$P_N(q(t,t') = q_n) = C_N(q_n,t)\exp(-N\alpha_t(q_n)) \tag{22}$$

where we have dropped the l dependence of the probability, which disappears at stationarity. Using Stirling approximation for the binomial coefficient and the saddle-point approximation to average over the distribution at time step $t-1$, we obtain the following equation for the evolution of the exponent of the distribution, $\alpha_t(x)$:

$$\alpha_t(x) = \alpha_{t-1}(q_t(x)) + x\log\left(\frac{x}{\gamma(q_t(x))}\right) + (1-x)\log\left(\frac{1-x}{1-\gamma(q_t(x))}\right) \tag{23}$$

where the function $q_t(x)$ must be determined self-consistently solving the equation

$$\alpha'_{t-1}(q_t(x)) - \gamma'(q_t(x))\left(\frac{x}{\gamma(q_t(x))} - \frac{1-x}{1-\gamma(q_t(x))}\right) \tag{24}$$

with the conditions $q_t(x) > 0$ and $q_t(x) < 1$.

At stationarity the most probable overlap (the point where $\alpha_t(q)$ has a minimum) is given by equation (21), and the variance of the distribution can be obtained by taking the second derivative of equation (23) and solving it together with the first derivative of the saddle-point condition (24). The result is

$$V^* = \frac{Q^*(1-Q^*)}{1-(\gamma'(Q^*))^2} = \frac{\frac{1}{4}}{1-(2/\pi)^2} \approx 0.4204 \tag{25}$$

where V^* is the variance of the stationary distribution multiplied by N. Thus, the variance is larger than in the case of a binomial distribution, since the dynamics has produced correlations between different elements.

The value of the closing probability cannot be computed analytically: for this we need the whole function $\alpha(x)$, and to obtain it we should solve a transcendent non-local equation. Thus we had to numerically solve equation (23), obtaining the stationary distribution reported in figure 1. The asymptotic closing probability, defined as $P^*_N(q=1) + P^*_N(q=0)$, is thus

$$\pi^*_N = 2\exp(\alpha N) \tag{26}$$

with $\alpha = 0.4554$. As discussed in the previous section, the exponent of the average length of the cycles should be equal to $\alpha/2$. This prediction is in good agreement with the numerical simulations that will be reported in section 4.

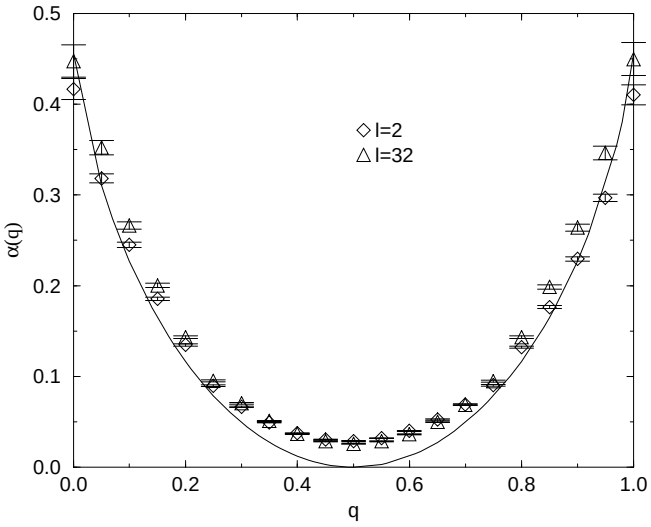

Figure 1. Exponent $\alpha(q)$ computed from the equation $P_N(q) = C_N(q)\exp(-N\alpha(q))$, where $P_N(q)$ is the stationary overlap distribution, with both $q = q(t, t+2)$ (diamonds) and $q = q(t, t+32)$ (triangles). The full curve shows the annealed prediction. The agreement is better in the second case, when the temporal distance between configurations is larger. The system size is $N = 20$ and $h = 0$.

2.4. Initial distribution

In order to compute the average number of cycles we have to know the distribution of the overlap with the initial configuration, $q(0, l)$, which plays the role of the initial distribution for the stochastic process studied in section 2.3. Although the number of cycles in fully asymmetric neural networks was exactly studied by Schreckenberg [11], we want to sketch the annealed computation of it, since it is much simpler and it can be generalized to more complex situations.

Our aim is to compute the distribution of $q(0, l)$. After one time step the annealed approximation is exact (we still have to extract all the couplings) and trivial: every spin can be either in its initial state or in the reversed one with a probability of $\frac{1}{2}$, and the overlap $q(0, 1)$ multiplied by N has a binomial distribution with $p = \frac{1}{2}$. After two time steps we distinguish two contributions in the local field: one coming from the set I_1 of the spins which are in the same state at $t = 0$ and at $t = 1$ and the other coming from all the other spins. We write

$$\sigma_i(0)\sigma_i(2) = \operatorname{sign}\left(\sum_{j\in I_1}\sigma_i(0)J_{ij}\sigma_j(0) - \sum_{j\in\Omega/I_1}\sigma_i(0)J_{ij}\sigma_j(0)\right). \tag{27}$$

Since the states $\sigma_i(0)$ and $\sigma_j(0)$ are independent both one on each other and on the couplings, we can set them equal to 1. If we change the sign of the last sum, we obtain $\sigma_i(0)\sigma_i(1)$. Thus, depending on whether this is positive or negative, there are two

possibilities:

$$\sigma_i(0)\sigma_i(2) = \begin{cases} \operatorname{sign}\left(\left(\sum_{j\in I_1} J_{ij}\right)^2 - \left(\sum_{j\in\Omega/I_1} J_{ij}\right)^2\right) & \text{if } \sigma_i(0)\sigma_i(1) > 0 \\ \operatorname{sign}\left(\left(\sum_{j\in\Omega/I_1} J_{ij}\right)^2 - \left(\sum_{j\in I_1} J_{ij}\right)^2\right) & \text{if } \sigma_i(0)\sigma_i(1) < 0 \end{cases} \tag{28}$$

(in this formula there is indeed a small imprecision, which becomes negligible in the infinite-size limit: since the coupling J_{ii} is set equal to zero, we do not have to take into account the spin $j = i$ itself, which contributes to the first sum in both lines). The probability that the sum of n Gaussian variables has a module larger than that of the sum of $N - n$ other Gaussian variables was computed in the previous section, where it received the name $\gamma(n/N)$. Taking all of this into account, we come to the transition probability

$$\begin{aligned}\Pr\{q(0,2) = m/N | q(0,1) = n/N\} \\ = \sum_k \binom{n}{k}\binom{N-n}{m-k}(\gamma(n/N))^{N-n-m+2k}(1-\gamma(n/N))^{n+m-2k}\end{aligned} \tag{29}$$

where, as usual, the opening condition imposes to exclude as starting points $n = 0$ and $n = N$, and the sum runs over all the values of k for which the factorial is well defined. The closing probability which can be deduced from this formula setting either $m = N$ or $m = 0$ coincides with the one exactly computed in [11]. It can easily be seen that it is proportional to $1/2^N$, as expected (the system loses memory of the initial configuration quite fast), and the proportionality coefficient can be computed with the saddle-point method [11].

In the general case, the information about $q(0, l)$ is not enough to compute the distribution of $q(0, l + 1)$: we also have to know the value of $q(0, 1)$, as it can be seen from equation (28) where we have to substitute 1 with l and 2 with $l + 1$ into the equations but we have to remember $\sigma_i(0)\sigma_i(1)$ in the conditions. In the general case the transition probability has thus the form

$$\begin{aligned}\Pr\{q(0,l+1) = m/N | q(0,1) = n_1/N, q(0,l) = n/N\} \\ = \sum_k \binom{n_1}{k}\binom{N-n_1}{m-k}(\gamma(n/N))^{N-n_1-m+2k}(1-\gamma(n/N))^{n_1+m-2k}\end{aligned} \tag{30}$$

and we have to consider the evolution of the joint distribution of the variables $q(0, 1)$ and $q(0, l)$. As expected, the correlations between these two variables vanish very fast as l grows, and the stationary distribution is the product of two binomial distributions, as it can be easily checked, so that for large l the closing probability is $\pi_N^{(a)}(0, l) = 1/2^N$ (with a equal to either 0 or 1), consistently with the supposed loss of memory and in agreement with the exact results of [11]. For small values of l it can be seen that $\pi_N^{(a)}(0, l) = c_l/2^N$, where c_l goes to a finite value in the infinite-size limit, so that the total number of cycles increases only proportionally to the system size.

3. Reversal symmetry

The computations shown in section 2.2 must be modified to take into account the twofold nature of the closing probability. We have to distinguish between two kinds of cycles, with different properties under the reversal operation.

(1) Cycles that close when $\mathcal{C}(t + l) = R\mathcal{C}(t)$ (or, in other words, $q(t, t + l) = 0$), whose length is $2l$. They are invariant under the reversal operation: each configuration is present together with its reversed one.

(2) Cycles that close when $q(t, t+l) = 1$. In this case the reversal operator applied to the cycle Γ produces a new cycle $R\Gamma$ with equal length and equally large attraction basin.

Taking this into account, we have to distinguish between cycles of even length, which can be of one of the two kinds, and cycles of odd length, which can only be of the first kind. Cycle length distribution is then

$$\Pr\{T = t, L = l\} = \frac{1}{2\tau^2}\exp\left(-\frac{(t+l)^2}{2\tau^2}\right) \qquad l \text{ odd} \tag{31}$$

$$= \frac{1}{2\tau^2}\exp\left(-\frac{(t+l)^2}{2\tau^2}\right) + \frac{1}{2\tau^2}\exp\left(-\frac{(t+l/2)^2}{2\tau^2}\right) \qquad l \text{ even} \tag{32}$$

with $\tau = 1/\sqrt{\pi}^* = 1/\sqrt{2}\exp(0.2277N)$.

The cycles of the first type have only even length, so that their number is half of the number of the cycles of the second type. Using the result of section 2.2 and summing up the contributions of both types of cycles we obtain, at the leading order in N,

$$\sum_l \langle n_a(l)\rangle \approx \tfrac{3}{2}\log\tau = \tfrac{3}{4}\alpha N \tag{33}$$

which is $\frac{3}{2}$ times larger than in a random map with the same closing probability.

The most important difference between attractors in asymmetric neural networks and in a random map involves the distribution of the attraction basins weights. Let us consider separately cycles of the first type and cycles of the second type (taking only one cycle to represent each pair of cycles of the second type). We then obtain the expression of the moments $\langle Y_n\rangle$ of the distribution of the weights:

$$\langle Y_n\rangle = \tfrac{1}{2}\left\langle \sum_{\alpha'} W_{\alpha'}^n + 2\sum_{\alpha''}(W_{\alpha''}/2)^n \right\rangle \tag{34}$$

where the sum over α' and α'' of the weights are both normalized to one. Under the hypothesis that each of the two sets of weights is distributed as in a random map, we obtain

$$\langle Y_n\rangle = \left(\frac{1}{2} + \frac{1}{2^n}\right)\langle Y_n\rangle_{RM} \tag{35}$$

or, using (10),

$$\langle Y_{n+1}\rangle = \frac{1}{2}\frac{(n!)^2}{(2n+1)!}(4^n + 2^n). \tag{36}$$

Thus the moments of the distribution of the weights are smaller than in the usual random map, for instance $\langle Y_2\rangle = \frac{1}{2}$ instead of $\frac{2}{3}$. These results are in very good agreement with numerical simulations.

To prove equation (35) let us recall that $\langle Y_n\rangle$ can be interpreted as the probability that n randomly chosen trajectories reach the same attractor. We can compute such a quantity by using the closing probabilities and following exactly the same lines as in [19], but we have to remember that a closing event has two different meanings: either a closure on an identical configuration ($q = 1$) or a closure on a reversed configuration ($q = 0$). Thus, not all of the events which represent the closure of the n trajectories, and whose probability is exactly $\langle Y_n\rangle_{RM}$, have the meaning that the trajectories will ultimately meet. If the first trajectory closes with $q = 0$ (this happens with a probability of $\frac{1}{2}$), its attraction basin also contains all of the reversed configurations, and the following $n-1$ trajectories which close on it will then go to the same attractor, regardless on how they close. In contrast, if the first trajectory closes with $q = 1$, the following $n-1$ trajectories also have to close with

$q = 1$ in order to go to the same attraction basin (if they close with $q = 0$, they go to the reversed basin). In this case, whose probability is again $\frac{1}{2}$, a closing event is equivalent to an asymptotic meeting of the n trajectories only with a probability of $1/2^{n-1}$. Equation (35) is thus proved.

3.1. Explicit symmetry breaking

The above picture of the distribution of the attraction basins weights is completely destroyed by the introduction of a magnetic field, however small, in the equations of motion (1), which restores the distribution typical of a random map. We considered the dynamic rules

$$\sigma_i(t+1) = \mathrm{sign}\left(\sum_j J_{ij}\sigma_j(t) - h \right). \tag{37}$$

The magnetic field h has the biological meaning of the threshold of activation of the neurons. In real neurons, such a non-zero threshold exists and can be different from one neuron to another one. In our simplified model, we take a threshold which is constant among the different neurons. Its introduction explicitly breaks the symmetry respect to the reversal of all the neural activities.

In the framework of the annealed approximation, the conditional probability that the activity of a neuron is the same in two different time steps is not more symmetric, i.e. $\gamma(1-q)$ is different from $1-\gamma(q)$ and $\gamma(1)$ increases very rapidly with respect to the above case, thus making a reversed closure very unlikely, while $\gamma(0)$ decreases. After a straightforward calculation we obtain

$$\gamma(q) = 1 - \frac{2}{\pi}\int_{\arcsin\sqrt{q}}^{\pi/2} \exp\left(-\frac{1}{2}(h/\sin t)^2\right) \mathrm{d}t. \tag{38}$$

For a large threshold the closing probability differs from 1 by a value that cancels very fast, as $\exp(-h^2/2)$.

The attractors of the first type (such that $\Gamma = R\Gamma$) are completely destroyed in this way, while attractors of the second type do not live in pairs any more, and the distribution of the weights is of the random map type.

4. Numerical results

4.1. Distribution of the overlap and closing probabilities

Our first aim was to compare the distribution of the overlap predicted by the annealed approximation with the same distribution in the quenched system. As we wrote, the analogy holds if we measure the overlap between configurations only along the trajectories that are not yet closed when we do the measurement. Under this condition, we computed the distribution of the overlap $q(t, t+l)$ for l fixed and t large enough to suppose that the distribution has attained stationarity.

The exponent $\alpha(q)$ of the distribution of the overlap is defined by the equation $P_N(q) = C_N(q)\exp(-N\alpha(q))$, where the factor $C_N(q)$, proportional to $1/\sqrt{N}$, comes from the Stirling expansion of the binomial coefficient. Thus we computed $\alpha(q)$ by using the formula

$$\alpha(q) = -\frac{1}{N}\left(\log(P_N(q)) + \frac{1}{2}\log N\right). \tag{39}$$

The logarithmic term must not be subtracted when q is equal to 0 or 1, because in this case the $1/\sqrt{N}$ factor is no more present in the expansion of the binomial coefficient, and so we did not consider it for $q = 0$ and 1, interpolating linearly between the two formulae for values of q between 0 and 0.1 and between 0.9 and 1. In such data analysis we neglect terms of order $1/N$ (there is also an unknown coefficient in the expression of the probability $P(q)$), and the agreement between the annealed prediction for $\alpha(q)$ and the quenched data, compared in figure 1, is then very satisfactory even for a system of such a small size (we considered $N = 20$). When l, the temporal distance between configurations, is small, there are some discrepancies (for instance, for $l = 2$ the quenched distribution is much broader than expected, and the closing probability is consequently much higher), but when l is large the agreement improves (in particular, the variance of the distribution and the exponents $\alpha(0)$ and $\alpha(1)$ of the closing probabilities coincide within the errors with the predicted values). This fact sustains our interpretation that the annealed approximation is valid when the temporal distance is large, so that the system has forgotten the details of its evolution [14, 21].

Next we measured the closing probability $\pi_N(t, t+l)$. figure 2 represents this quantity as a function of t for different values of l, kept fixed. The statistic errors are large, but it appears that $\pi(t, t+l)$ reaches a value approximately stationary in t, in agreement with the annealed prediction, when l is large (in figure 2(b) we have $l = 11$) but when l is small (in figure 2(a) $l = 2$) the closing probability reaches a maximum value and then decreases, as a function of t. We already observed this kind of non-monotonic behaviour of the closing probabilities in simulations of Kauffman model. In both cases we interpret the decreasing part of $\pi(t)$ as due to the opening condition: the condition that the trajectory is not closed at time t selects, as t grows, trajectories that are more and more unlikely to close. The opening condition cannot be imposed in the annealed scheme, because we consider the stochastic process $d(t, t+l)$ with l fixed and we cannot control $d(t, t')$ for generic t and t'. So the annealed scheme must be modified to take this fact into account [14]. But in asymmetric neural networks, differently from what we observed in Kauffman networks, the opening condition seems to be irrelevant when the temporal distance l is large, and the closing probability appears to reach, in this case, an approximately stationary value.

The non-stationarity of the distribution of $q(t, t+2)$ shows the existence of memory effects in the model: the statistical properties of $q(t, t+2)$ still depend on t, even after an arbitrarily long transient time. It would be interesting to find out whether the lack of time translation invariance in the system with completely uncorrelated couplings has some relation with aging in the relaxational dynamics of the SK spin glass model [22]. In the present case, however, the lack of time translation invariance is only a minor effect and does not prevent the overlap $q(t, t+l)$ from reaching a stationary distribution for l large enough. The macroscopic properties of the dynamics can be predicted, in good agreement with numerical results, also neglecting this effect at all.

We conclude this section by showing a plot of the integral closing probability $\tilde{\pi}_N(t)$, defined as

$$\tilde{\pi}_N(t) = \sum_{t'=0}^{t-1} \pi_N(t', t) \tag{40}$$

this is the probability that a trajectory not closed at time $t-1$ closes at time t). In Kauffman networks, this quantity is non-monotonic as a function of t: it increases to a maximum value and then decreases with t. On the other hand, from the annealed approximation we would expect it to increase linearly with t in the stationary state. In asymmetric neural networks we found that the integral closing probability increases monotonically with t. After a transient

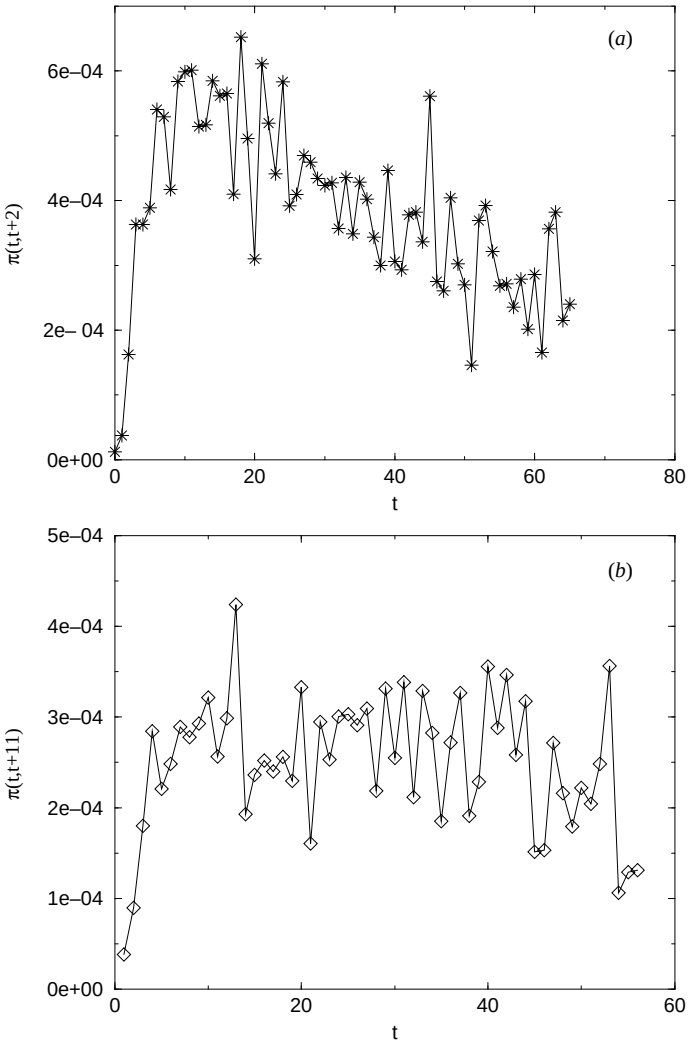

Figure 2. Closing probability $\pi_N(t, t+l)$ as a function of t, for (a) $l = 2$ and (b) $l = 11$. The system size is $N = 20$ with $h = 0$.

phase of very fast increase it slows down, and asymptotically it appears to behave as a power law. For the largest systems that we simulated our data are very noisy, and we could fit the asymptotic t behaviour only for $N = 20$, finding that the best fit exponent of the power law is approximately 0.6. So, at least for systems of not very large size, deviations from the annealed approximation are also present in this case.

4.2. Properties of attractors: Zero threshold

To obtain the first three moments of the distribution of the attraction basins we followed the method indicated in [20]. For every value of the parameters N and $h = 0$ we generated at random 2000 networks, extracting the synaptic couplings with Gaussian distribution, and we simulated four randomly chosen trajectories on each of them.

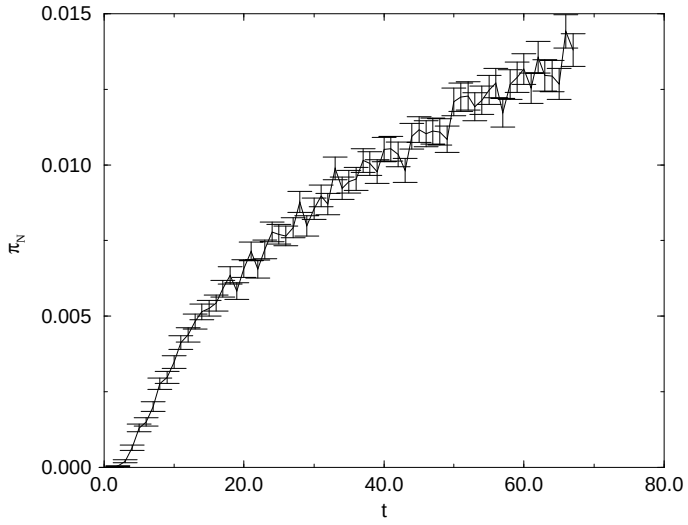

Figure 3. Integral closing probability, $\pi_N(t) = \sum_{t'} \pi_N(t', t)$ as a function of t in a system of size 20 with $h = 0$.

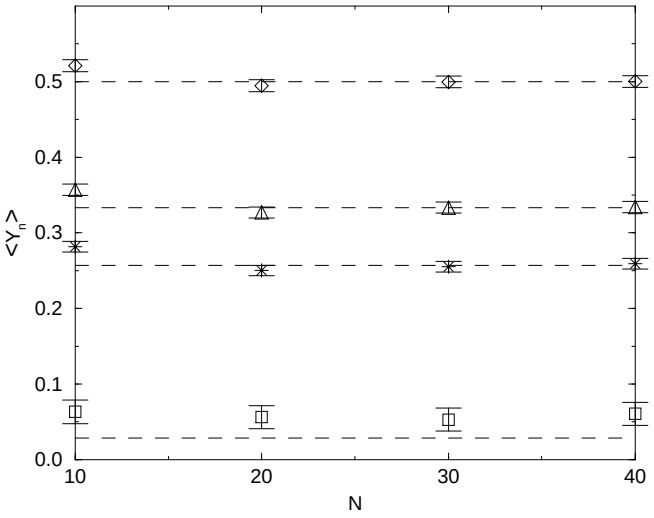

Figure 4. Moments of the distribution of the attraction basin weights versus system size N for $h = 0$: $\langle Y_2 \rangle$ (diamonds); $\langle Y_2^2 \rangle$ (squares); $\langle Y_3 \rangle$ (triangles) and $\langle Y_4 \rangle$ (stars). The dotted lines show the predictions of the annealed approximation.

The average weight of the basins, $\langle Y_2 \rangle$, was estimated from the probability that two different trajectories end up on the same periodic orbit. In general [20], $\langle Y_n \rangle$ can be measured as the probability that n different initial configurations evolve to the same attractor. Simulating four initial configurations it is also possible to measure $\langle Y_2^2 \rangle$ as the probability that each of two pairs of configurations end up on a same attractor, the two attractors being either different or equal.

Figure 4 shows data which report the behaviour of the moments of attraction basin distribution for systems of different size, N. It can be seen that they rapidly converge to

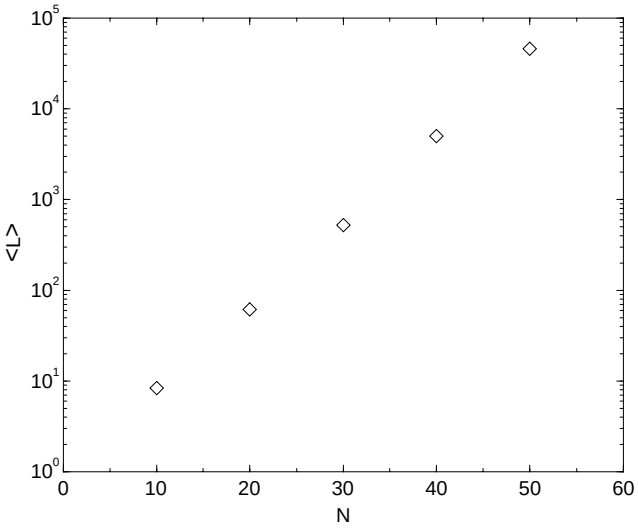

Figure 5. Average length of the cycles as a function of system size for $h = 0$.

the predictions of the annealed approximation, corrected to take into account the reversal symmetry.

The average cycles length increases exponentially with system size, $\langle L \rangle \propto \exp(\alpha_L/2N)$. The exponent $\alpha_L/2$ is less than $\log 2/2 = 0.347$, as it would be in a completely random map. Its value $\alpha_L/2 = 0.224$ is in good agreement with the prediction of the annealed approximation, $\alpha/2 = 0.228$ (the small discrepancy could be a finite-size effect, as the exponent estimated from numerical data increases when only the largest systems are considered). Figure 5 shows the average length of the cycles versus system size.

The distribution of cycle length is much broader than expected on the basis of the annealed approximation, and asymptotically behaves as a stretched exponential:

$$\Pr\{L > l\} \approx \exp(-(l/\tau_N)^{\gamma_N}). \tag{41}$$

As discussed in the previous sections, the distribution is different for the two different types of cycles. We considered only odd cycles, in order to select only attractors of the second type, and we checked that the scale of the distribution, τ_N, increases exponentially with N, $\tau_N \propto \exp(\alpha_P/2N)$, where the exponent α_P coincides with α_L within the errors. On the other hand, the exponent γ_N of the stretched exponential, for which the annealed approximation predicts the value 2, is instead less than 1 for all of the system sizes that we examined, but it appears to increase slightly as N grows (though are data about this point are very noisy), so that it is possible that this discrepancy shall disappear in the infinite-size limit.

The fact that we find γ_N to be less than 1 also appears challenging because the distribution of the closing time (i.e. the sum of the transient time plus the length of the cycle), which should have the same behaviour of the distribution of cycle length, according to the annealed approximation, is indeed much steeper: it can be fitted to a stretched exponential of the same form (41), but with a much larger exponent γ'_N. For instance, for $N = 20$, we find $\gamma'_N = 1.9$, in good agreement with the annealed prediction, while the value of γ_N is 0.69.

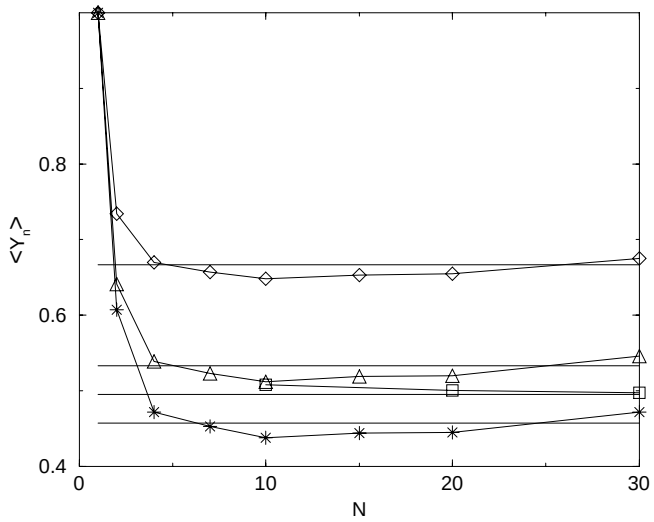

Figure 6. Moments of the distribution of the attraction basin weights versus system size N for $h = 0.1$: $\langle Y_2 \rangle$ (diamonds); $\langle Y_2^2 \rangle$ (squares); $\langle Y_3 \rangle$ (triangles) and $\langle Y_4 \rangle$ (stars). The dotted lines show the predictions of the annealed approximation.

4.3. Properties of attractors: Broken symmetry

When we consider the evolution equation (37) with a threshold h, the reversal symmetry is explicitly broken and the distribution of the weights is the same as in the usual random map.

Figure 6 shows the behaviour of the first moments of the distribution of the weights as a function of system size N for $h = 0.1$. Such a threshold is so small that it modifies the value of the exponent α by less than 2%: the annealed approximation predicts in this case $\alpha = 0.448$, to be compared with the value of 0.455 found with zero threshold. The prediction is in good agreement with numerical simulations: a fit of the average length of the cycles gives $\alpha_L = 0.44$. For such a small threshold we can observe traces of the broken symmetry present as finite-size effects: the moments of the distribution at small N fall below the random map values, even if of a very small amount, and then increase to those values, which are maintained asymptotically in system size.

On the other hand, when the threshold is larger, we do not see at all the signs of the symmetry on the distribution: for $h = 1$, the average basin weight decreases monotonically from the value 1 at small N towards the random map value $\langle Y_2 \rangle = \frac{2}{3}$. For such a threshold the average length of the cycles still behaves exponentially with N, but the exponent α is very small and power law corrections also have important effects for systems large to simulate, as it appears from the fact that the best fit exponent depends significantly on system size (it decreases as system size increases), and we could not estimate it accurately. Nevertheless, the agreement between the annealed approximation, which predicts $\alpha = 0.128$, and the numerical result $\alpha_L = 0.15$, is worse than in the previous cases but is still not bad.

5. Summary and conclusions

In this work we used a stochastic scheme, based on the closing probabilities and on their approximation by means of a Markovian stochastic process, in order to compute

the properties of the attractors in fully asymmetric neural networks. The fundamental hypothesis behind this approximation is that the system forgets fast enough the details of its past evolution, so that a one-step memory is already enough to describe the gross features of the dynamics. Our method is able to predict very satisfactorily the N behaviour of the typical lengths of the cycles and typical transient times, the number of cycles, the distribution of their attraction basin weights and also the main features of the distribution of the distances. On the other hand, the approximation fails to predict the shape of the distribution of cycle length, which is much broader than we would expect.

The average number of cycles has already been exactly computed, and perhaps other quantities can be exactly computed in this model, but the present method has the advantage of being very simple, and we hope that it can be applied to more complex situations. In particular, with this method we argue that the distribution of attraction basins is, for disordered dynamical systems that are 'chaotic enough', always equal to the one computed by Derrida and Flyvbjerg [19] for the case of a uniform random map.

A possible extension of our method, that we consider to be very interesting and that we plan to pursue further, is towards the study of neural networks with finite symmetry. Numerical studies suggest that such systems undergo an abrupt change of dynamical regime when the symmetry η is changed [9], but a 'mean field' description of this transition from an ordered behaviour to chaos is still lacking. The possibility that such a change can be characterized as a transition between memory and loss of memory is very appealing. Memory effects are more and more important for networks with non-zero coupling symmetry (until the well known aging properties of the SK model are approached). Because of these effects, it is necessary to modify our method to also study the chaotic regime of the model (low symmetry). Technically this is not an easy task, since non-zero symmetry correlations arise both between the local fields (see equations (18)) of different neurons and, more difficult to treat, between synaptic couplings and dynamical variables. The latter introduce an effective interaction between the state of an element at two different time steps t and $t+2$, so that in the dynamics also an effective gradient flow is present, and, if the annealed approximation can describe this situation, it will be necessary to take into account also this information, aside the crude distance, to make the annealed scheme useful.

Studying this family of models it is also possible, by varying the continuous parameters η and h, to go from the distribution of the attraction basins typical of the random map to the one typical of spin glasses, thus the study of the general model would shed some light on the relation between the two kinds of distributions.

Memory effects are probably responsible of the discrepancy between the prediction of the annealed approximation and the observed distribution of cycle length. In fact, the distance $d(t, t')$ does not reach a stationary distribution, if we impose the condition that the trajectory is not yet closed before the measure. We think that this condition, which cannot be imposed in our computation, selects trajectories that are less and less likely to close. As a result, the integral closing probability, which is our main tool in the computation, increases as a function of t slower than expected. It is possible that this effect shows up only at small l and disappears in the infinite-size limit (a suggestion of this could be the fact that the distribution of cycle length decays faster in this limit), but it is also possible that, as in the case of the Kauffman model that we previously studied, some corrections to the annealed picture are necessary also in the infinite-size limit. However, we think that these results show that the annealed approximation is a useful tool to investigate the properties of attractors in disordered dynamical systems in a simple way.

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